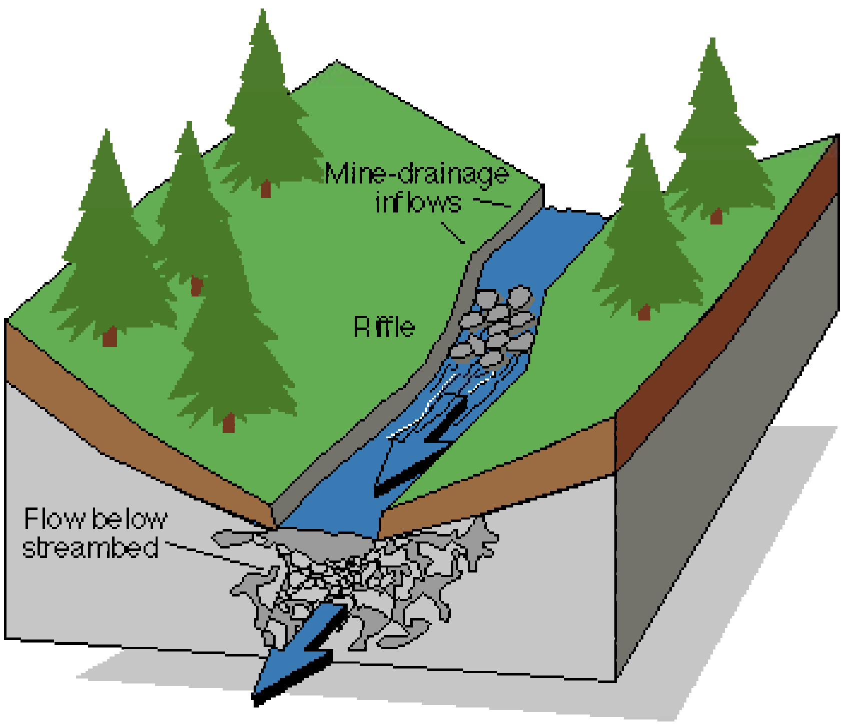
Boundary Conditions at a Stream Aquifer Interface



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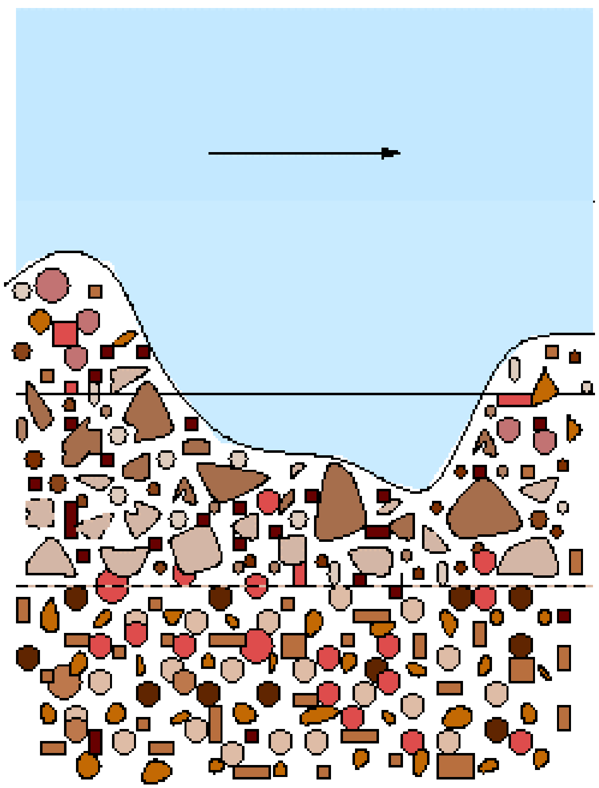
Averaging Up From Pore-Scale



- Navier-Stokes

$$\nabla \bullet \vec{U} = 0$$

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U} \bullet \nabla)\vec{U} + g\nabla P - \frac{\mu}{\rho}\nabla^2 \vec{U} = 0$$

- Length Scales
  - L_d=pore diameter
  - L_{avg}=averaging length
  - L_a=aquifer depth
  - $L_d << L_{avg} << L_a$

Macro-Scale Equations

Averaging Function: $\vec{V} = \Omega^{-1} \int_{\Omega_f} \vec{U} \partial \Omega_f$

- **Stream equations**

$$\nabla \bullet \vec{V}_s = 0, \quad \frac{\partial \vec{V}_S}{\partial t} + (\vec{V}_S \bullet \nabla)\vec{V}_S + g\nabla P_S - \frac{1}{\rho}\vec{M} = 0$$

- **Interfacial equations**

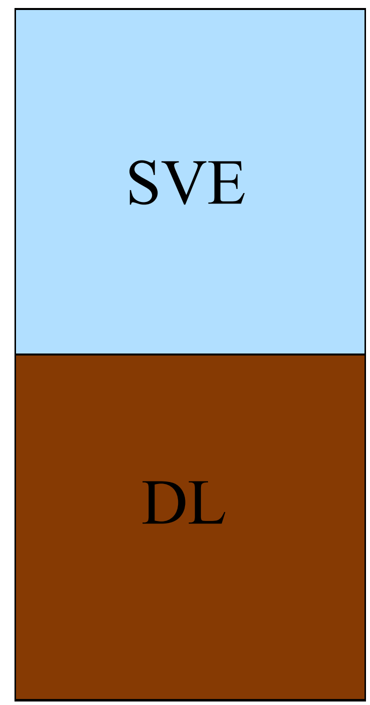
$$\nabla \bullet \vec{V}_i = 0, \quad \frac{\partial \vec{V}_i}{\partial t} + (\vec{V}_i \bullet \nabla)\vec{V}_i + g\nabla(P_i\theta) - \frac{\mu}{\rho}\nabla^2\vec{V}_i = -gK^{-1}\vec{V}_i + \nabla \bullet \vec{D}$$

- **Aquifer equations**

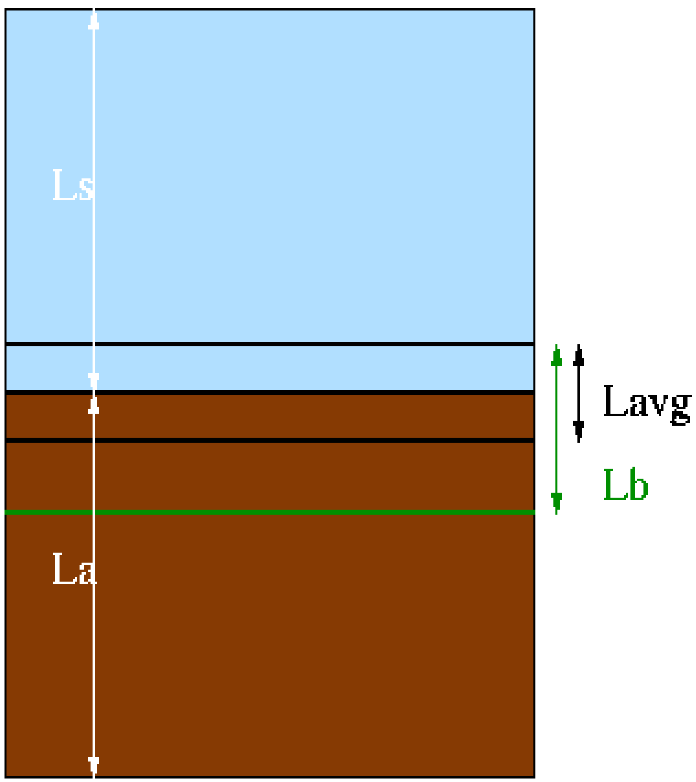
$$\nabla \bullet \vec{V}_a = 0, \quad \theta_0 \nabla P_a = -K^{-1}\vec{V}_a$$

Mass and Momentum Transfer Across the Stream-Aquifer Interface

- $\vec{V}^s, P^s$ solve SVE for $z > z_{sb}$
- $\vec{V}^a, P^a$ solve DL for $z < z_{sb}$
- Boundary conditions follow from requiring the variables to satisfy the interfacial equations for mass and momentum transfer in a weak sense (on average) near the interface



Separation of Length Scales



- L_{avg} macroscopic average length scale
- L_s characteristic depth of stream
- L_a characteristic depth of aquifer
- L_b depth of viscous "Brinkman layer"
- L_∞ large scale average
- $L_{avg} << L_b < L_\infty < L_a$

Derivation of BC's

Let $\varphi\vec{V}^a + (1-\varphi)\vec{V}^s$ solve $\int_{\Omega_\infty} (\nabla \bullet \vec{V})\partial\Omega = 0$.

Then,

$$\int_{\Omega_\infty} \{\varphi\nabla \bullet \vec{V}^a + (1-\varphi)\nabla \bullet \vec{V}^s + \nabla\varphi \bullet (\vec{V}^a - \vec{V}^s)\}\partial\Omega = 0$$

Since $\nabla \bullet \vec{V}^a = 0$ for $z<z_{sb}$, and $\nabla \bullet \vec{V}^s = 0$ for $z>z_{sb}$, then

$$\int_{\Omega_\infty} \nabla\varphi \bullet \{\vec{V}^a - \vec{V}^s\}\partial\Omega = \int_{\Omega_\infty} \frac{\partial\varphi}{\partial z}\{V_3{}^a - V_3{}^s\}\partial\Omega = 0$$

Derivation of BC's

Let $\varphi \vec{V}^a + (1-\varphi)\vec{V}^s$ solve

$$\int_{\Omega_\infty} \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \bullet \nabla)\vec{V} + g\nabla P - \frac{\mu}{\rho}\nabla^2 \vec{V} - \nabla \bullet \vec{D} + gK^{-1}\vec{V}\right)\partial\Omega = 0$$

Then, we obtain:

$$\int_{\Omega_\infty} \nabla\varphi \bullet \{(V_j^a \vec{V}^a - V_j^s \vec{V}^s) + g\theta(P^a - P^s)\vec{I}_j - \frac{\mu}{\rho}(\nabla V_j^a - \nabla V_j^s)\}\partial\Omega$$

$$= \int_{\Omega_\infty} 2gK^{-1}V_j^a \widetilde{H}(z)\partial\Omega + O(L_a)$$

Boundary Conditions

At the stream-aquifer interface, $(x,y,z)=(x,y,z_{sb})$:

$$V_3^a = V_3^s$$

$$-gL_b K^{-1} V_1^a = V_3^a (V_1^s - V_1^a) + \frac{\mu}{\rho}\frac{\partial V_1^a}{\partial z} + \frac{gn^2}{L_b^{1/3}} V_1^s \,|\vec{V}^s|$$

$$-gL_b K^{-1} V_2^a = V_3^a (V_2^s - V_2^a) + \frac{\mu}{\rho}\frac{\partial V_2^a}{\partial z} + \frac{gn^2}{L_b^{1/3}} V_2^s \,|\vec{V}^s|$$

$$-gL_b K^{-1} V_3^a = g\theta(P^s - P^a) + \frac{\mu}{\rho}\left(\frac{\partial V_3^a}{\partial z} - \frac{\partial V_3^s}{\partial z}\right)$$

Generalized Boundary Conditions

At the stream-aquifer interface, $(x,y,z)=(x,y,z_{sb}(x,y))$:

$$\vec{V}^a \bullet \vec{n} = \vec{V}^s \bullet \vec{n}$$

$$-gL_b K^{-1}\vec{V}_j^a = g\theta(P^s - P^a)(\vec{I}_j \bullet \vec{n}) + \frac{\mu}{\rho}(\nabla \vec{V}_j^a - \vec{M}) \bullet \vec{n}$$

Where the momentum flux term at the interface is

$$\vec{M} = \left(\frac{\rho g n^2}{\mu L_b^{1/3}} V_1^s \,|\vec{V}^s|, \frac{\rho g n^2}{\mu L_b^{1/3}} V_2^s \,|\vec{V}^s|, \frac{\partial V_3^s}{\partial z}\right)$$